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It is shown that the two nonessential but removable singularities of the plane circular restricted three-body problem can be removed simultaneously by a coordinate transformation, which is defined by the conformal mapping

$$Z = \frac{1}{4}\left(\zeta^n + \frac{1}{\zeta^n}\right)$$

where n is any finite and nonzero real number. Another regularization is obtained by the mapping

$$Z = \frac{1}{2} \cos n\, \zeta$$

which is related to the preceding one by substituting $e^{i\zeta}$ for ζ.

The first mapping gives Birkhoff's transformation when $n = 1$. But when n has the value 2, this transformation is related to a coordinate system which has been introduced by G. Lemaitre in the general three-body problem. This transformation for $n = 2$ is also essentially equivalent to the regularization introduced by R. F. Arenstorf. The second transformation gives Thiele's well-known regularization when n takes the value $+1$.

Author

I. INTRODUCTION

The study of the motion of three points with finite masses, which attract each other according to the Newtonian gravitation law, is equivalent to the resolution of a system of second-order differential equations where the unknowns are the coordinates of the three bodies. But it seems that these differential equations have in the denominator the third powers of the distances r_1, r_2, or r_3 between the masses. For this reason, the equations are always valid except for the singularities $r_1 = 0$, $r_2 = 0$, or $r_3 = 0$. In fact, a detailed examination shows that the three-body problem has two kinds of singularities with a completely different behavior. The first kind of singularity corresponds to the collision of two of the three bodies. In this collision, only one of the three distances, r_1, r_2, r_3, becomes zero. But there is one other singularity in the three-body problem, corresponding to the simultaneous collision of the three bodies. In this collision the three distances r_1, r_2, r_3 vanish together. The three first singularities are called binary collisions, while the other is the triple collision.

The main difference between the two kinds of singularities is that the triple collision is an essential singularity which cannot be removed by coordinate transformations, while the binary collisions are removable singularities. It is possible to introduce coordinate transformations such that the binary collision singular points disappear.

The present study is confined to the restricted three-body problem which describes the motion of a satellite under the action of two masses subject to their Keplerian motion around the center of mass. It is supposed here that the orbits of the two main masses are circles. In the restricted three-body problem, there are not four, but only two singularities corresponding to the collision of the satellite with one of the two masses. Both singularities are removable singularities, as has been shown first by Thiele's regularization about 70 years ago (Ref. 1).

In the existing literature concerning the restricted three-body problem, essentially three fundamental regularizing coordinate systems have been well described in the last 70 years. The three different methods can all be expressed by conformal mappings.

Thiele, in his work, introduces two new variables E, F, which can be defined from the rectangular coordinates X, Y by the conformal mapping:

$$Z = X + iY$$

$$\theta = E + iF$$

$$Z = \frac{1}{2}\cos\theta$$

This transformation removes the two singularities together, and gives remarkably simple forms for the equations, or for the Lagrangian and Hamiltonian. The major difficulties with Thiele's coordinates appear when they are used for electronic numerical calculations. Several trigonometric and other transcendental functions must be evaluated, and the calculations become longer, while cumulative errors appear very easily. However, Thiele's coordinates have been used during more than 30 years by the calculators of Stromgren's group at the Copenhagen Observatory in order to find about a hundred periodic orbits by hand calculations (Ref. 2).

In 1904, Levi-Civita introduced a new regularization of the circular restricted three-body problem (Ref. 3), by the parabolic coordinates which can be defined by the simple mapping:

$$Z = X + iY$$

$$\zeta = \xi + i\eta$$

$$Z = \zeta^2$$

The parabolic coordinates have the property that they cannot remove the two singularities together, but only one, which must be located at the origin of the X, Y coordinate system. The parabolic coordinates lead to simple differential equations, without transcendental functions, and for this reason the author used the parabolic coordinates in order to calculate 6,000 orbits with a digital electronic computer (Ref. 4).

In 1915, Birkhoff defined a new regularization (Ref. 5), which can be represented by a conformal mapping:

$$Z = \frac{1}{4}\left(\zeta + \frac{1}{\zeta}\right)$$

The two singularities are removed together, but not many authors seem to have used Birkhoff's coordinates.

Recent work has shown that Birkhoff's coordinates can be generalized (Ref. 6, 7) by a conformal mapping:

$$Z = \frac{1}{4}\left(\zeta^2 + \frac{1}{\zeta^2}\right)$$

These new coordinates are related to works done by Lemaitre in the general three-body problem (Ref. 8), and for this reason the author has called them Lemaitre's coordinates. This coordinate system gives differential equations which are distinguished by the fact that they are expressed by purely algebraic forms in the coordinates and their canonical moments, without square roots and transcendental functions.

It has been shown by Wintner in Ref. 9 that the parabolic regularization cannot be generalized in

$$Z = \zeta^n$$

where n is any integer. Only the value $n = 2$ is useful for the purpose of regularization. In this Report, it is shown, however, that Thiele's coordinates and Birkhoff's coordinates can be generalized in

$$Z = \frac{1}{2}\cos n\,\theta$$

$$Z = \frac{1}{4}\left(\zeta^n + \frac{1}{\zeta^n}\right)$$

where n is any nonzero real number. We obtain thus two one-parameter families of regularizing coordinate transformations. These two families of coordinate transformations are strongly related. In order to see the relationship, it is sufficient to define a supplementary mapping:

$$\zeta = e^{i\theta}$$

This Report develops some of the properties of the different conformal mappings, and gives the Hamiltonian function for the restricted three-body problem with the regularizing variables. However, once Birkhoff's regularization is known, then it is obvious that, for any real nonzero number n, the generalized transformation also gives a regularization, for the simple reason that the substitution $\zeta \rightarrow \zeta^n$ is regular anywhere (except for $n = 0$, $\zeta = 0$ and $\zeta = \infty$, but the generality is not restricted if these special values are excluded). Although the regularization is made for any real (nonzero) value of the parameter n, the case where n is an integer number is the most interesting. The conversion from the physical coordinates to the regularized coordinates is more simple when n is an integer number, because the separation of $Z(\zeta)$ in the real and the imaginary part becomes more simple. But in fact, the separation of $Z(\zeta)$ in the real part and the imaginary part is not the important question for the purpose of numerical computations. In the application of Birkhoff's generalized regularization for the numerical integration of the restricted three-body problem, one arbitrary determination of ζ should be taken for starting the integration, and after the integration should be transformed back to the unique determination of Z.

In Section VII, it is shown that there is, for $n = 2$, a remarkable relation with the general three-body problem, by a stereographic projection which has been introduced by Lemaitre (Ref. 8).

II. THE GENERAL EQUATIONS FOR THE PLANE CIRCULAR RESTRICTED THREE-BODY PROBLEM

It is well known that the plane circular restricted three-body problem can be referred to a uniformly rotating *barycentric* coordinate system, with a choice of units such that the equations of motion are

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x = -\frac{\partial V}{\partial x}$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y = -\frac{\partial V}{\partial y}$$

where

$$V = -\left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)$$

$$r_1^2 = (x - x_1)^2 + y^2$$

$$r_2^2 = (x - x_2)^2 + y^2$$

$$x_1 = -\mu$$

$$x_2 = 1 - \mu$$

The preceding equations of motion are derived from the Lagrangian

$$L = \frac{1}{2}(x'^2 + y'^2) - (yx' - xy') + \frac{1}{2}(x^2 + y^2) - V$$

and have the energy, or Jacobi, integral

$$E = \frac{1}{2}(x'^2 + y'^2) - \frac{1}{2}(x^2 + y^2) + V$$

By a simple translation along the x-axis,

$$x = X + x_0,$$

$$y = Y,$$

$$x_0 = \frac{1}{2} - \mu$$

we can also refer the problem to another rotating coordinate system X, Y, where the origin is now the middle of both main masses $1-\mu$ and μ, instead of being the center of mass. The distance between the center of mass and the middle of the masses is x_0, and for this reason the corresponding coordinates X, Y are called *median* coordinates. The Lagrangian can then be written in the form:

$$L = \frac{1}{2}(X'^2 + Y'^2) - (X'Y - XY') + \frac{1}{2}(X^2 + Y^2) - V + x_0 X$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (Yp_x - Xp_y) + V - x_0 X$$

The coordinates of the two main masses are now $(-1/2, 0)$ and $(+1/2, 0)$, and the distance r_1 and r_2 are given by

$$r_1^2 = \left(X + \frac{1}{2}\right)^2 + Y^2$$

$$r_2^2 = \left(X - \frac{1}{2}\right)^2 + Y^2$$

Starting from the coordinate system X, Y, some regularizing coordinate systems shall now be introduced by conformal mappings.

III. GENERAL PROPERTIES OF THE CONFORMAL MAPPING $Z = Z(\zeta)$ APPLIED TO THE RESTRICTED THREE-BODY PROBLEM

Let Z and ζ be the complex numbers

$$Z = X + iY$$

$$\zeta = \xi + i\eta$$

Then the analytic function

$$Z = Z(\zeta)$$

defines a conformal mapping between the two complex planes Z and ζ, except for the critical points of the mapping. At the regular points we have the Cauchy-Riemann equations

$$\frac{\partial X}{\partial \xi} = \frac{\partial Y}{\partial \eta}$$

$$\frac{\partial X}{\partial \eta} = -\frac{\partial Y}{\partial \xi}$$

and the Jacobian J may be written in one of the following forms:

$$J = \begin{vmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial X}{\partial \eta} \\ \frac{\partial Y}{\partial \xi} & \frac{\partial Y}{\partial \eta} \end{vmatrix} = \left|\frac{dZ}{d\zeta}\right|^2 = \frac{1}{4}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)(X^2 + Y^2)$$

or

$$J = \left(\frac{\partial X}{\partial \xi}\right)^2 + \left(\frac{\partial X}{\partial \eta}\right)^2 = \left(\frac{\partial Y}{\partial \xi}\right)^2 + \left(\frac{\partial Y}{\partial \eta}\right)^2$$

$$= \left(\frac{\partial X}{\partial \xi}\right)^2 + \left(\frac{\partial Y}{\partial \xi}\right)^2 = \left(\frac{\partial X}{\partial \eta}\right)^2 + \left(\frac{\partial Y}{\partial \eta}\right)^2$$

We shall now apply to the equations of the restricted three-body problem a coordinate transformation given by the preceding general conformal mapping. The Hamiltonian with the new variables ξ, η and their corresponding canonical moments p_ξ, p_η takes the general form

$$H = \left|\frac{dZ}{d\zeta}\right|^{-2} \left[\frac{1}{2}(p_\xi^2 + p_\eta^2) - (A_\xi p_\xi + A_\eta p_\eta) + \left|\frac{dZ}{d\zeta}\right|^2 \tilde{V}(\xi, \eta)\right]$$

where

$$A_\xi = -\frac{1}{2}\frac{\partial}{\partial \eta}(X^2 + Y^2)$$

$$A_\eta = +\frac{1}{2}\frac{\partial}{\partial \xi}(X^2 + Y^2)$$

$$\tilde{V} = V - x_0 X$$

If we now introduce a new independent variable s, related with the time variable t by the relation

$$dt = \left|\frac{dz}{d\zeta}\right|^2 ds$$

we obtain a new Hamiltonian

$$\overline{H} = \frac{1}{2}(p_\xi^2 + p_\eta^2) - (A_\xi p_\xi + A_\eta p_\eta) + \left|\frac{dZ}{d\zeta}\right|^2 (\widetilde{V}(\xi, \eta) - h)$$

where h is the energy constant corresponding to the energy integral $H = h$.

The Lagrangian equations of motion may then be written with the new variables ξ, η, s, in the form

$$\ddot{\xi} - 2\left|\frac{dz}{d\zeta}\right|^2 \dot{\eta} = -\frac{\partial \overline{V}}{\partial \xi}$$

$$\ddot{\eta} + 2\left|\frac{dZ}{d\zeta}\right|^2 \dot{\xi} = -\frac{\partial \overline{V}}{\partial \eta}$$

where

$$\overline{V} = \left|\frac{dz}{d\zeta}\right|^2 (\widetilde{U}(\xi, \eta) + h)$$

$$= \left|\frac{dZ}{d\zeta}\right|^2 \left[\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}(X^2 + Y^2) + x_o X + h\right]$$

The convention used here is that a prime always indicates a derivative with respect to the time t, while a dot indicates a derivative with respect to the new regularized time s.

IV. THE CONFORMAL MAPPING $Z = 1/4\ (\zeta^n + \zeta^{-n})$

We suppose that n is a finite and nonzero real number. The definition of the conformal mapping

$$Z = \frac{1}{4}\left(\zeta^n + \frac{1}{\zeta^n}\right)$$

can be written in the form of a second-degree equation in ζ^n,

$$\zeta^{2n} - 4Z\,\zeta^n + 1 = 0$$

and by solving this equation for ζ^n we find that

$$\zeta^n = 2\left[Z \pm \sqrt{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{2}\right)}\right]$$

$$= F(Z) = X_1 + iY_1$$

Thus, when n is not integer or rational, there are generally two infinite sets of points ζ corresponding to each point Z. Each set corresponds to one of the signs of the square root. Each of these sets is on a circle with the origin of the coordinate system as center. One of these circles has a radius greater than 1, while the other has a radius smaller than 1. All these facts are obvious consequences of elementary considerations of the complex function ζ of the complex variable Z. All the possible determinations of ζ are complex numbers with constant moduli

$$|\zeta| = |F|^{1/n}$$

and with arguments,

$$\arg(\zeta) = \frac{1}{n}\arg(F) + \frac{2k\pi}{n}$$

$$\pm k = 0, 1, 2, \cdots$$

There is an exception when the square root is zero; i.e., when Z corresponds to one of the singularities $m_1(Z = -1/2)$ or $m_2(Z = +1/2)$ of the problem. For these two points of the Z-plane there is only one infinite set of corresponding points in the ζ-plane. For both singularities, the ζ-points are on the unit circle. The arguments are $2k\pi/n$ for m_2 and $(2k+1)\,\pi/n$ for m_1.

It is easy to describe now the correspondence between the Z-plane and the ζ-plane. To the segments $X < -1/2$, $Y = 0$, and $1/2 < X, Y = 0$, there are corresponding rectilinear segments in the ζ-plane, all passing by the

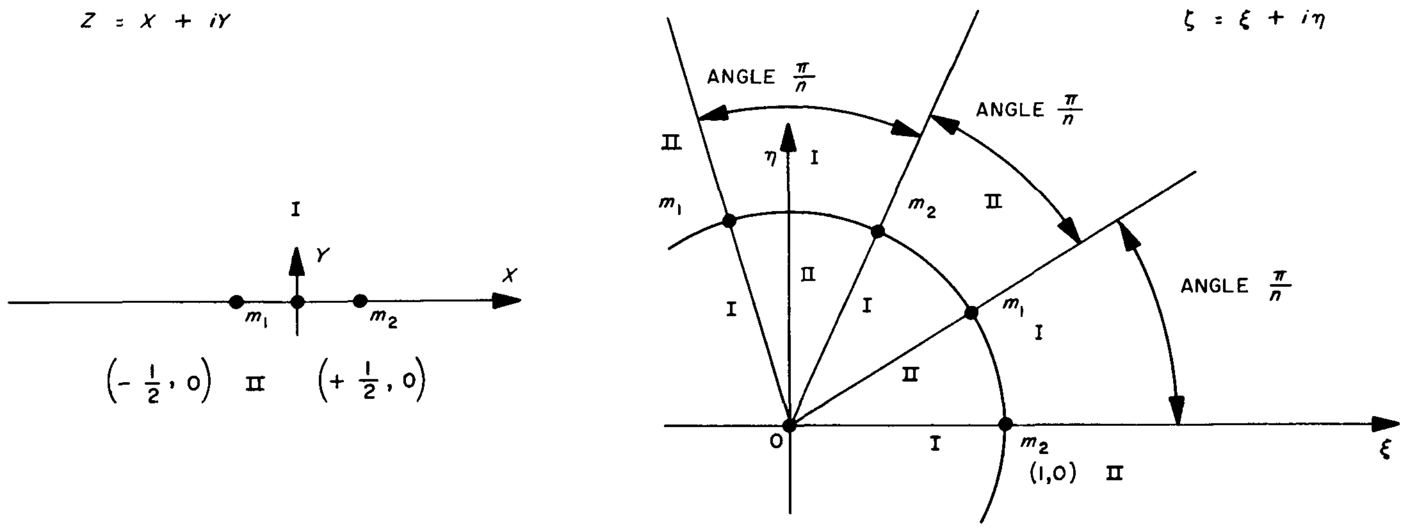

Fig. 1. Conformal mapping $Z = 1/4\ (\zeta^n + 1/\zeta^n)$,

origin. To the segment $-1/2 < X < +1/2$, $Y = 0$, there are corresponding arcs of circles with unit radius in the ζ-plane. Each half Z-plane corresponds to sectors of angle π/n in the ζ-plane. The correspondence is illustrated in Fig. 1.

The most relevant fact is the separation of the real and imaginary parts of Z. By introducing S according to

$$S = \xi^2 + \eta^2 = \zeta\,\bar{\zeta} = |\,\zeta\,|^2$$

we can write the conformal mapping in the form:

$$Z = \frac{1}{4S^n}\,(S^n\,\zeta^n + \bar{\zeta}^n)$$

and S^n has the value

$$S^n = X_1^2 + Y_1^2 = (\xi^2 + \eta^2)^n$$

We can now write

$$X = \frac{1}{4S^n}\,(S^n + 1)\,X_1$$

$$Y = \frac{1}{4S^n}\,(S^n - 1)\,Y_1$$

and also

$$X^2 + Y^2 = \frac{1}{16}\left[\left(S^n + \frac{1}{S^n}\right) + \frac{2}{S^n}\left(X_1^2 - Y_1^2\right)\right]$$

When n is an integer or rational number, a few simplifications occur for this conformal mapping. We do not consider here the case where n is rational, but not integer. When n is integer, the conformal mapping sets up 1-to-$2n$ correspondence between the Z-plane and the ζ-plane. There are here also only two exceptions, where we have a 1-to-n correspondence at

$$m_1 = (1 - \mu)$$

where $Z = -\dfrac{1}{2}$,

and at

$$m_2 = \mu$$

where $Z = +\dfrac{1}{2}$.

Corresponding to m_1, we have $\zeta^n = -1$; i.e., n points in the complex ζ-plane which are the n nth roots of -1, or n points on the unit circle with arguments $\pi\,(2k + 1)/n$ for $k = 0, 1, \cdots, n - 1$. Corresponding to m_2, we have $\zeta^n = +1$; i.e., the n complex nth roots of the number $+1$. We also have n points on the unit circle which have the arguments $\pi 2k/n$, for $k = 0, 1, \cdots, n - 1$. Corresponding to $z = \infty$, we have $\zeta = 0$ or ∞, but we always restrict ourselves to finite points in the z- and ζ-planes, and to nonzero points in the ζ-plane. When n is integer, X_1 and Y_1 are both homogeneous polynomials of degree n in ξ and η. They may be expressed in terms of the binomial coefficients C_n^i:

$$X_1 = \xi^n - C_n^2\,\eta^2\,\xi^{n-2} + C_n^4\,\eta^4\,\xi^{n-4} - C_n^6\,\eta^6\,\xi^{n-6} + \cdots$$

$$Y_1 = C_n^1\,\eta\,\xi^{n-1} - C_n^3\,\eta^3\,\xi^{n-3} + C_n^5\,\eta^5\,\xi^{n-5} - \cdots$$

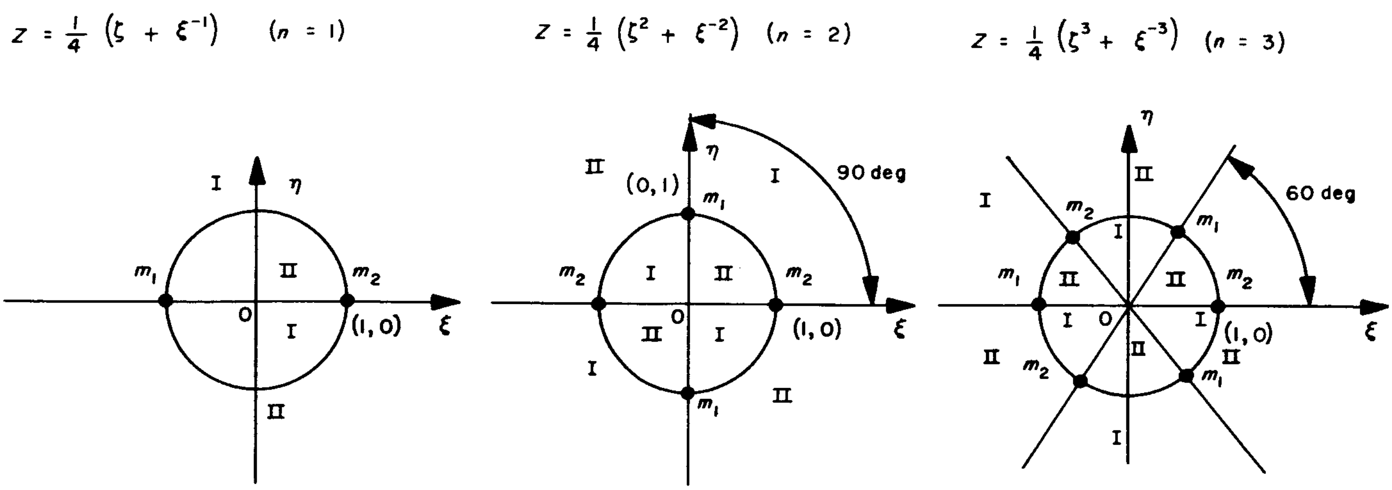

Fig. 2. Conformal mapping $Z = 1/4\,(\zeta^n + 1/\zeta^n)$ for $n = 1, 2, 3$

For instance, when $n = 1$, we have

$$X_1 = \xi_1$$

$$Y_1 = \eta$$

and when $n = 2$,

$$X_1 = \xi^2 - \eta^2$$

$$Y_1 = 2\,\xi\eta$$

The simple cases $n = 1$, 2, and 3 are illustrated in Fig. 2.

In the next developments we shall need the Jacobian of our transformation. For this purpose we have

$$\left(\frac{dZ}{d\zeta}\right)^2 = \frac{n^2}{16\zeta^2}\,[\zeta^{2n} + \zeta^{-2n} - 2]$$

and, on the other side,

$$r_1\,r_2 = \left|Z^2 - \frac{1}{4}\right| = \frac{1}{16}\left|\zeta^{2n} + \zeta^{-2n} - 2\right|$$

Thus we arrive at the interesting form for the Jacobian

$$J = \left|\frac{dZ}{d\zeta}\right|^2 = \frac{r_1\,r_2\,n^2}{S}$$

We arrive thus at the conclusion that our coordinate transformation has only two critical points, corresponding to $r_1 = 0$ and $r_2 = 0$; i.e., the two singularities of the restricted three-body problem we want to regularize. They are also coincident with the two branching points.

The Jacobian can also be expressed directly as a function of ξ and η, in the form

$$J = \frac{n^2}{16S^{n+1}}\,[(S^n + 1)^2 - 4X_1^2]$$

because we can verify directly that

$$r_1 = \frac{1}{4\sqrt{S^n}}\,[(S^n + 1) - 2X_1]$$

$$r_2 = \frac{1}{4\sqrt{S^n}}\,[(S^n + 1) + 2X_1]$$

and then also that

$$r_1\,r_2 = \frac{1}{16S^n}\,[(S^n + 1)^2 - 4X_1^2]$$

V. THE NEW HAMILTONIAN OF THE RESTRICTED THREE-BODY PROBLEM

We now have enough results to write the Hamiltonian of the restricted three-body problem in terms of the new variables ξ, η. We have the Hamiltonian and the energy equation

$$H = \frac{1}{r_1 r_2}\left\{\frac{S}{2n^2}\cdot(p_\xi^2 + p_\eta^2) - \frac{S}{n^2}\cdot(A_\eta\, p_\eta + A_\eta\, p_\eta) - [(1-\mu)\, r_1 + \mu\, r_2] - x_0\, X r_1\, r_2\right\} = h$$

By the introduction of the new time variable s,

$$dt = r_1\, r_2\, ds$$

we obtain the new Hamiltonian

$$\overline{H} = \frac{S}{2n^2}\cdot(p_\xi^2 + p_\eta^2) - \frac{S}{n^2}(A_\xi\, p_\xi + A_\eta\, p_\eta) - [(1-\mu)\, r_1 + \mu\, r_2] - (x_0\, X + h)\, r_1\, r_2$$

and the corresponding canonical equations

$$\frac{d\xi}{ds} = \frac{\partial \overline{H}}{\partial p_\xi}$$

$$\frac{dp_\xi}{ds} = -\frac{\partial \overline{H}}{\partial \xi}$$

$$\frac{d\eta}{ds} = \frac{\partial \overline{H}}{\partial p_\eta}$$

$$\frac{dp_\eta}{ds} = -\frac{\partial \overline{H}}{\partial \eta}$$

It is essential to see that the preceding Hamiltonian and the associated equations of motion have no singularities. For this reason we say that they are regularized. The most interesting particular cases are $n = 1$ and $n = 2$. The value $n = 1$ gives Birkhoff's coordinates, and the case $n = 2$ is related to Lemaitre's investigations in the field of the general three-body problem.

Summarized below are the most important relations which are sufficient for the numerical integration of the equation of motion, in the Lagrangian, as well as in the Hamiltonian form, with the special values $n = 1$ and 2. It is remarkable to see that the equations, for $n = 2$, are not only regularized, but also that they do not contain square roots to be evaluated. This is true not only for $n = 2$, but for all the even integer values of n.

The explicit real form for the coordinate transformation is given by the relations

$n = 1$	$n = 2$
$X_1 = \xi$	$X_1 = \xi^2 - \eta^2 = D$
$Y_1 = \eta$	$Y_1 = 2\xi\eta$
$X = \frac{\xi}{4}\left(1 + \frac{1}{S}\right)$	$X = \frac{D}{4}\left(1 + \frac{1}{S^2}\right)$
$Y = \frac{\eta}{4}\left(1 - \frac{1}{S}\right)$	$Y = \frac{\xi\eta}{2}\left(1 - \frac{1}{S^2}\right)$

For both the Lagrangian and the Hamiltonian we need the explicit expression of $|dZ/d\zeta|^2$. The explicit expressions in ξ, η are

$n = 1$	$n = 2$
$r_1 = \frac{1}{4\sqrt{S}}[S + 1 + 2\xi]$	$r_1 = \frac{1}{4S}[S^2 + 2D + 1]$
$r_2 = \frac{1}{4\sqrt{S}}[S + 1 - 2\xi]$	$r_2 = \frac{1}{4S}[S^2 - 2D + 1]$
$r_1\, r_2 = \frac{1}{16S}[S^2 - 2D + 1]$	$r_1\, r_2 = \frac{1}{16S^2}[(S^2 + 1)^2 - 4D^2]$
$\left\lvert\frac{dZ}{d\zeta}\right\rvert^2 = \frac{1}{16S^2}[S^2 - 2D + 1]$	$\left\lvert\frac{dZ}{d\zeta}\right\rvert^2 = \frac{1}{4S^3}[(S^2 + 1)^2 - 4D^2]$

The potential term $\bar{V}$ can be written in the form

$$\bar{V} = \left|\frac{dz}{d\zeta}\right|^2 (V - x_0 X\text{-}h) = \frac{n^2}{S}\left[(1-\mu) r_2 + \mu r_1 - r_1 r_2 (x_0 X + h)\right]$$

where all the variable quantities in the right-side member are known functions of ξ, η.

Finally, for the Hamiltonian, the two quantities A_ξ, A_η are given by the relations

$n = 1$	$n = 2$
$X^2 + Y^2 = \frac{1}{16S}\left[S^2 + 1 + 2D\right]$	$X^2 + Y^2 = \frac{1}{16S^2}\left[(S^2 - 1)^2 + 4D^2\right]$
$A_\xi = -\frac{\eta}{16S^2}\left[S^2 - 4\xi^2 - 1\right]$	$A_\xi = -\frac{\eta}{8S^3}\left[(S^4 - 1) - 8D\,\xi^2\right]$
$A_\eta = \frac{\xi}{16S^2}\left[S^2 + 4\eta^2 - 1\right]$	$A_\eta = \frac{\xi}{8S^3}\left[(S^4 - 1) + 8D\,\eta^2\right]$

The most important partial derivatives which have to be used in the equations of motion are

$n = 1$	$n = 2$
$\frac{\partial X}{\partial \xi} = \frac{\partial Y}{\partial \eta} = \frac{1}{4}\left(1 - \frac{D}{S^2}\right)$	$\frac{\partial X}{\partial \xi} = \frac{\partial Y}{\partial \eta} = \frac{-\xi}{2S^3}\left[2D - S(S^2 + 1)\right]$
$\frac{\partial X}{\partial \eta} = -\frac{\partial Y}{\partial \xi} = \frac{-\xi\eta}{2S^2}$	$\frac{\partial X}{\partial \eta} = -\frac{\partial Y}{\partial \xi} = \frac{-\eta}{2S^3}\left[2D + S(S^2 + 1)\right]$
$\frac{\partial r_1}{\partial \xi} = \frac{1}{4\sqrt{S^3}}\left[(S - 1)\xi + 2\eta^2\right]$	$\frac{\partial r_1}{\partial \xi} = \frac{\xi}{2S^2}\left[S^2 + 4\eta^2 - 1\right]$
$\frac{\partial r_1}{\partial \eta} = \frac{\eta}{4\sqrt{S^3}}\left[(S - 1) - 2\xi\right]$	$\frac{\partial r_1}{\partial \eta} = \frac{\eta}{2S^2}\left[S^2 - 4\xi^2 - 1\right]$
$\frac{\partial r_2}{\partial \xi} = \frac{1}{4\sqrt{S^3}}\left[(S - 1)\xi - 2\eta^2\right]$	$\frac{\partial r_2}{\partial \xi} = \frac{\xi}{2S^2}\left[S^2 - 4\eta^2 - 1\right]$
$\frac{\partial r_2}{\partial \eta} = \frac{\eta}{4\sqrt{S^3}}\left[(S - 1) + 2\xi\right]$	$\frac{\partial r_2}{\partial \eta} = \frac{\eta}{2S^2}\left[S^2 + 4\xi^2 - 1\right]$
$\frac{\partial (r_1 r_2)}{\partial \xi} = \frac{\xi}{8S^2}\left[S^2 - 4\eta^2 - 1\right]$	$\frac{\partial (r_1 r_2)}{\partial \xi} = \frac{\xi}{4S^3}\left[S^4 - 8D\,\eta^2 - 1\right]$
$\frac{\partial (r_1 r_2)}{\partial \eta} = \frac{\eta}{8S^2}\left[S^2 + 4\xi^2 - 1\right]$	$\frac{\partial (r_1 r_2)}{\partial \eta} = \frac{\eta}{4S^3}\left[S^4 + 8D\,\xi^2 - 1\right]$

VI. THE CONFORMAL MAPPING $Z = 1/2 \cos n\theta$

The coordinate transformation $Z = 1/4\,(\zeta^n + \zeta^{-n})$ takes an interesting form if we introduce a new transformation, with complex variables

$$\zeta = e^{i\theta}$$

or, in the real form,

$$\xi = e^{-F} \cos E$$

$$\eta = e^{-F} \sin E$$

$$E = \operatorname{arctg} \frac{\eta}{\xi}$$

$$F = -\frac{1}{2} \log_e (\xi^2 + \eta^2)$$

where θ is the complex variable:

$$\theta = E + iF$$

Thus we see that E, F are a special system of polar coordinates in the (ξ, η)-plane; e^{-F} is a radius, while E is an angle. If we limit E to the limits $0, 2\pi$, then we have a 1-to-1 correspondence between the ζ-plane and the θ-plane.

With the introduction of the θ-variable, the Z-to-ζ correspondence now becomes a Z-to-θ correspondence, which can be written in the form

$$Z = \frac{1}{2} \cos n\,\theta$$

The real form of this conformal mapping is very easy to obtain, and is not restricted to the integer values of n, because we have

$$\zeta^n = e^{-nF} (\cos nE + i \sin nE)$$

$$X_1 = e^{-nF} \cos nE$$

$$Y_1 = e^{-nF} \sin nE$$

with $Y_1^2 + Y_1^2 = S^n = e^{-2nF}$.

The real form of our mapping thus is

$$X = \frac{1}{2} \cos nE \operatorname{ch} nF$$

$$Y = -\frac{1}{2} \sin nE \operatorname{sh} nF$$

The following simple formulas are useful for several computations:

$$r_1 = \frac{1}{2} (\cos nE + \operatorname{ch} nF) = \cos^2 \frac{nE}{2} + \operatorname{sh}^2 \frac{nF}{2}$$

$$r_2 = \frac{1}{2} (-\cos nE + \operatorname{ch} nF) = \sin^2 \frac{nE}{2} + \operatorname{sh}^2 \frac{nF}{2}$$

$$r_1 r_2 = \frac{1}{4} (\operatorname{ch}^2 nF - \cos^2 nE)$$

$$2 \sin^2 \frac{nE}{2} = (-r_1 + r_2 + 1)$$

$$2 \cos^2 \frac{nE}{2} = (+r_1 - r_2 + 1)$$

$$2 \operatorname{sh}^2 \frac{nF}{2} = (+r_1 + r_2 - 1)$$

$$2 \operatorname{ch}^2 \frac{nF}{2} = (+r_1 + r_2 + 1)$$

On the other hand, E, F can easily be obtained as functions of r_1, r_2 or X, Y by the formulas

$$r_1 + r_2 = \operatorname{ch} nF$$

$$r_1 - r_2 = \cos nE$$

The last two formulas show that the ellipses $r_1 + r_2 =$ constant in the z-plane correspond to straight lines $F =$ constant in the θ-plane, while the hyperbolas $r_1 - r_2 =$ constant correspond to the straight lines $E =$ constant. When the integer n takes the particular value 1, we have Thiele's well-known regularizing coordinates.

We still have two infinite sets of values E, F for each pair X, Y as far as n is not integer, nor rational. If E_0, F_0 is one couple corresponding to X, Y, then all such couples are

$$\left(E_0 + \frac{2k\pi}{n}, F_0\right) \text{ and } \left(-E_0 + \frac{2k\pi}{n}, -F_0\right)$$

where $\pm k = 0, 1, 2, \cdots$.

The Jacobians of the transformations are

$$\left|\frac{dZ}{d\zeta}\right|^2 = \frac{r_1 r_2 n^2}{S}$$

$$\left|\frac{d\zeta}{d\theta}\right|^2 = e^{-2F} = S$$

$$\left|\frac{dZ}{d\theta}\right|^2 = \left|\frac{dz}{d\zeta}\right|^2 \cdot \left|\frac{d\zeta}{d\theta}\right|^2 = n^2 r_1 r_2$$

The Hamiltonian can now be written in the form:

$$H = \frac{1}{n^2 r_1 r_2}\left[\begin{array}{l} 2\,(p_E^2 + p_F^2) - (\mathrm{sh}\, nF\, \mathrm{ch}\, nF \cdot p_E + \sin nE \cos nE \cdot p_F) \\ + 2\,(2\,x_0 \cos nE - \mathrm{ch}\, nF) \;\; - 2\,x_0\, r_1\, r_2 \cos nE \cdot \mathrm{ch}\, nF \end{array}\right]$$

A detailed study of the particular case $n = 1$ has been given in Ref. 6. The relationship between Thiele's and Birkhoff's generalized coordinates is illustrated in Fig. 3.

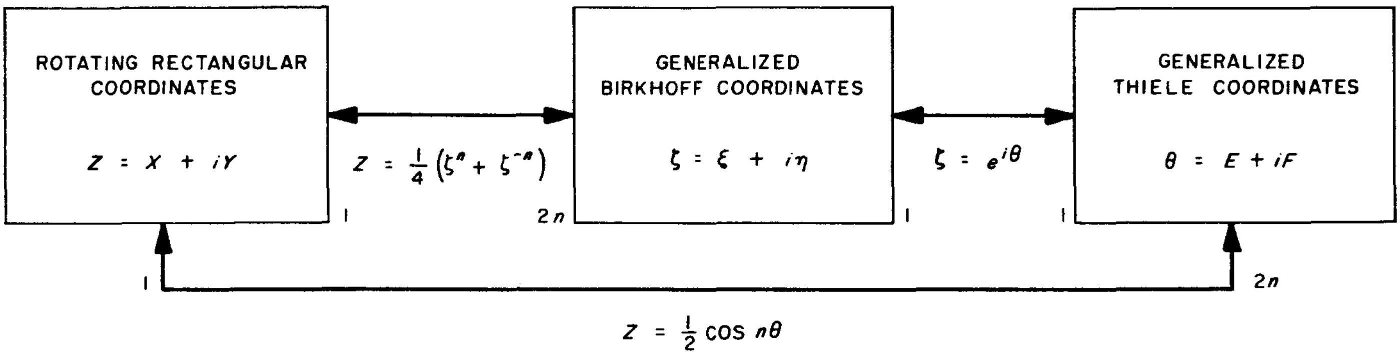

Fig. 3. Relations between rotating rectangular coordinates, Birkhoff's coordinates and Thiele's coordinates

VII. RELATIONS WITH THE GENERAL THREE-BODY PROBLEM (REF. 8-10)

Murnaghan (Ref. 10) obtained symmetric equations for the general three-body problem by using the three distances r_1, r_2, r_3 as the principal variables. But the binary collision singularities were still present in these equations. Lemaitre (Ref. 8) removed all three singularities together by introducing three new variables q_1, q_2, q_3, instead of the distances r_1, r_2, r_3, as follows:

$$2\,q_1^2 = (-r_1 + r_2 + r_3)$$

$$2\,q_2^2 = (+\,r_1 - r_2 + r_3)$$

$$2\,q_3^2 = (+\,r_1 + r_2 - r_3)$$

Lemaitre also defined

$$2\,q^2 = (+\,r_1 + r_2 + r_3) = 2\,(q_1^2 + q_2^2 + q_3^2)$$

In this way, we can write

$$r_1 = q_2^2 + q_3^2 = q^2 - q_1^2$$

$$r_2 = q_1^2 + q_3^2 = q^2 - q_2^2$$

$$r_3 = q_1^2 + q_2^2 = q^2 - q_3^2$$

Lemaitre also proposed to make a stereographic projection in the three-dimensional space q_1, q_2, q_3 of the point $(q_1/q,\ q_2/q,\ q_3/q)$ into the plane q_1, q_2 from the South Pole (0, 0, -1). The coordinates of the projection point in the q_1, q_2 plane are then

$$\xi = \frac{q_1}{q + q_3}$$

$$\eta = \frac{q_2}{q + q_3}$$

and we have also the reciprocal relations

$$q_1 = q\left(\frac{2\xi}{(1+S)}\right)$$

$$q_2 = q\left(\frac{2\eta}{(1+S)}\right)$$

$$q_3 = q\left(\frac{(1-S)}{(1+S)}\right)$$

with $S = \xi^2 + \eta^2$.

It is interesting now to see how this formulation behaves when applied to the circular restricted three-body problem. In this problem we have at each instant $r_3 = 1$.

The preceding relations thus give

$$q_1^2 + q_2^2 = 1$$

$$q^2 - q_3^2 = 1$$

and we see that the circular restricted three-body problem is represented in the (q_1, q_2, q_3)-space by a vertical cylinder with the q_3-axis as the symmetry axis and with the radius + 1.

We shall now see what the relation is between the coordinates X, Y of the satellite in the rectangular Cartesian coordinate system and the stereographic projection ξ, η of the corresponding point of the cylinder. We have seen that

$$r_1^2 = \left(X + \frac{1}{2}\right)^2 + Y^2$$

$$r_2^2 = \left(X - \frac{1}{2}\right)^2 + Y^2$$

or that

$$X = \frac{1}{2}(r_1^2 - r_2^2)$$

$$Y^2 = \frac{1}{4}\left[(r_1^2 + r_2^2) - (r_1^2 - r_2^2)^2 - 1\right]$$

These last two relations give X and Y as functions of q, q_1, q_2, q_3:

$$X = -\frac{1}{2}(q^2 + q_3^2)(q_1^2 - q_2^2)$$

$$Y = 2q\, q_1 q_2 q_3$$

or as functions of ξ, η:

$$X = -\frac{D}{4S^2}(1 + S^2)$$

$$Y = \frac{\xi\eta}{2S^2}(1 - S^2)$$

where $D = \xi^2 - \eta^2$, and $S = \xi^2 + \eta^2$.

The last two relations may now be written in the complex form

$$Z = X + iY$$

$$\zeta = \xi + i\eta$$

$$Z = \frac{1}{4}\left(\zeta^2 + \frac{1}{\zeta^2}\right)$$

and we see the identity of Lemaitre's coordinates and the generalized Birkhoff coordinates, where the integer n takes the value 2.

We can also establish other relationships between the generalized Thiele coordinates E, F and q, q_1, q_2, q_3; for instance,

$$q_1 = \sin\frac{nE}{2}$$

$$q_2 = \cos\frac{nE}{2}$$

$$q_3 = \mathrm{sh}\frac{nF}{2}$$

$$q = \mathrm{ch}\frac{nF}{2}$$

These relations show that the curves E = constant are vertical straight lines on the cylinder, and that the curves F = constant are the latitude circles on the same cylinder in the (q_1, q_2, q_3)-space.

VIII. CONCLUSIONS

This Report gives a synthesis of a few important regularizations of the plane circular restricted three-body problem. The regularizations which have been given by Thiele, Birkhoff, Lemaitre, and Arenstorf are described in a unified form, by using conformal mappings. However, Levi-Civita's regularization is not included, and many other regularizing coordinates could be introduced, for instance, with canonical transformations.

All the regularizing conformal mappings that have been described have the common property that they have a critical point which is at the same time a branch point, corresponding to the regularized singularity of the three-body problem.

The Jacobian of the transformation has in all the regularizations one of the three following forms:

1. $J = A(\xi, \eta)\, r_1$, with $A(\xi, \eta) \neq 0$ for $r_1 = 0$
2. $J = A(\xi, \eta)\, r_2$, with $A(\xi, \eta) \neq 0$ for $r_2 = 0$
3. $J = A(\xi, \eta)\, r_1 r_2$, with $A(\xi, \eta) \neq 0$ for $r_1 r_2 = 0$

They correspond respectively to the three situations:

1. The collision $r_1 = 0$ is regularized only.
2. The collision $r_2 = 0$ is regularized only.
3. Both collisions $r_1 = 0$ and $r_2 = 0$ are regularized simultaneously.

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